A TRANSVERSAL FREDHOLM PROPERTY FOR THE $\bar\partial$-NEUMANN PROBLEM ON G-BUNDLES

DEDICATED TO M.A. SHUBIN ON HIS 65TH BIRTHDAY

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ABSTRACT. Let M be a strongly pseudoconvex complex G-manifold with compact quotient M/G. We provide a simple condition on forms α sufficient for the regular solvability of the equation $\Box u = \alpha$ and other problems related to the $\bar\partial$-Neumann problem on M.

1. Introduction

Let M be a manifold which is the total space of a G-bundle

$$G \longrightarrow M \longrightarrow X$$

with X compact. With respect to a G-invariant measure on M, define the Hilbert space $L^2(M)$. This decomposes as

$$L^2(M) \cong L^2(G) \otimes L^2(X), \tag{1.1}$$

and if we assume that the action of G is from the right, then $t \in G$ acts in $L^2(M)$ by $t \to R_t \otimes \mathbf{1}_{L^2(X)}$. The von Neumann algebra of operators on $L^2(G)$ commuting with right translations is denoted by $\mathcal{L}_G$ and the corresponding algebra of bounded linear operators on $L^2(M)$ that commute with the action of G is denoted by $\mathcal{B}(L^2(M))^G$. This has a decomposition itself as follows,

$$\mathcal{B}(L^2(M))^G \cong \mathcal{B}(L^2(G) \otimes L^2(X))^G \cong \mathcal{L}_G \otimes \mathcal{B}(L^2(X))).$$

Definition 1.1. Let M be a G-manifold with quotient $X = M/G$ and let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces of sections of bundles over M. A closed, densely defined, linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ which commutes with the action of G is called *transversally Fredholm* if the following conditions are satisfied:

(1) there exists a finite-rank projection $P_{L^2(X)} \in \mathcal{B}(L^2(X))$ such that $\ker A \subset \operatorname{im}(\mathbf{1}_{L^2(G)} \otimes P_{L^2(X)})$
(2) there exists a finite-rank projection $P'_{L^2(X)} \in \mathcal{B}(L^2(X))$ such that $\operatorname{im} A \supset \operatorname{im}(\mathbf{1}_{L^2(G)} \otimes P'_{L^2(X)})^\perp$.

This note will provide a simple example of this idea. Let M be a strongly pseudoconvex complex manifold which is also the total space of a G-bundle $G \longrightarrow M \longrightarrow X$ with X compact. Furthermore, assume that G acts on M by holomorphic transformations. With respect to a G-invariant measure and Riemannian structure, define the Hilbert spaces of (p, q)-forms $L^2(M, \Lambda^{p,q})$.

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On M, consider Kohn's Laplacian, $\Box$ and its spectral decomposition, $\Box = \int_0^\infty \lambda dE_\lambda$ in $L^2(M,\Lambda^{p,q})$. If $q>0$, it was shown in [P1] that if $\delta \geq 0$, then the Schwartz kernel of the spectral projection $P_\delta = \int_0^\delta dE_\lambda$ belongs to $C^\infty(\bar{M}\times\bar{M})$. Choosing a piecewise smooth section $X \hookrightarrow M$, we may write points in M as pairs $(t,x)\in G\times X$. The Schwartz kernel K of P_δ then, almost everywhere, takes the form

$$K(t,x;s,y) = K(ts^{-1},x;e,y) =: \kappa(ts^{-1};x,y),$$

where we have used the G-invariance of P_δ. It is also true that κ has an expansion

$$\kappa(t;x,y) = \sum_{kl} \psi_k(x) h_{kl}(t)\bar{\psi}_l(y) \tag{1.2}$$

where $(\psi_k)_k$ is an orthonormal basis of $L^2(X)$. The functions h_{kl} are smooth in G with $\sum_{kl}\|h_{kl}\|^2_{L^2_R(G)}<\infty$, where $L^2_R(G)$ consists of the functions on G that are square-integrable with respect to right-Haar measure (*cf.* proof of Lemma 6.2 in [P1]).

The main result of the present paper is the fact that when κ corresponds to P_δ, the sum in equation (1.2) can be taken to be finite. This means that the spectral projections of $\Box$ are subordinate to simple projections of the form $P = \mathbf{1}_{L^2(G)}\otimes P_{L^2(X)}$ with $P_{L^2(X)}$ the projection onto the space spanned by the ψ_k that appear in the sum. Since there are finitely many, we have that $\operatorname{rank} P_{L^2(X)}<\infty$. Thus our main result in this note is

Theorem 1.2. *Let M be a strongly pseudoconvex complex manifold which is also the total space of a G-bundle $G \longrightarrow M \longrightarrow X$ with X compact. Furthermore, assume that G acts on M by holomorphic transformations. It follows that for $q>0$, the Laplacian $\Box$ in $L^2(M,\Lambda^{p,q})$ is transversally Fredholm.*

We will also show that the $\bar{\partial}$-Neumann problem has regular solutions for $g\in \operatorname{im} P^\perp$.

As well as sharpening the results in [P1], the results of this note will be useful in studying the $\bar{\partial}$-Neumann problem and its consequences for G-manifolds with nonunimodular structure group; in [P1], G was always assumed unimodular. These G-manifolds, among others, occur naturally as complexifications of group actions, as shown in [HHK].

The present results, in addition to the amenability property introduced in [P2], will lead to a better understanding of two important exemplary nonunimodular G-manifolds discussed in [GHS]. One of these has a large space of L^2-holomorphic functions while the other has $L^2\mathcal{O}=\{0\}$.

Remark 1.3. All the results in this note remain valid for weakly pseudoconvex M satisfying a subelliptic estimate, and for the boundary Laplacian, $\Box_b$, [P3].

2. Invariant operators in $L^2(M)$

Here we briefly sketch the construction of the Schwartz kernel (1.2) of P_δ. We will continue to simplify notation by suppressing the operators' acting in bundles; some additional details are in [P1].

On the group alone, the projection P_L onto a translation-invariant subspace $L\subset L^2(G)$ is a left-convolution operator with distributional kernel κ,

$$(P_L u)(t) = (\lambda_\kappa u)(t) = \int_G ds\ \kappa(ts^{-1})u(s), \qquad (u\in L^2(G)),$$

where ds is the right-invariant Haar measure.

Let us lift this definition to $L^2(M)$ by taking the decomposition (2) a step further. Letting $(\psi_k)_k$ be an orthonormal basis for $L^2(X)$, we may write

$$L^2(M) \cong L^2(G) \otimes L^2(X) \cong \bigoplus_k L^2(G) \otimes \psi_k,$$

and with respect to this decomposition write matrix representations for operators in $L^2(M)$ as

$$\mathcal{B}(L^2(M)) \ni P \longleftrightarrow [P_{kl}]_{kl}, \qquad P_{kl} \in \mathcal{B}(L^2(G)).$$

When $P \in \mathcal{B}(L^2(M))^G$ each of the P_{kl} is an operator commuting with the right action and thus is a left convolution operator. Thus $P_{kl} = \lambda_{h_{kl}}$ for distributions h_{kl} on G, as in the expansion (1.2). When P is a self-adjoint projection, we find that the matrix of convolutions $H = [\lambda_{h_{kl}}]_{kl}$ is an idempotent in that $\sum_k H_{jk}H_{kl} = H_{jk}$ and the matrix corresponding to P^*, has matrix representation $[\lambda^*_{h_{lk}}]_{kl}$.

3. Regularity of the $\bar\partial$-Neumann problem on G-manifolds

We provide a brief list of the properties of the $\bar\partial$-Neumann problem relevant to our work here and refer the reader to [FK, GHS, P1] for more detail. With the invariant measure and Riemannian structure on M define the Sobolev spaces $H^s(M, \Lambda^{p,q})$ of (p,q)-forms on M. Note that the G-invariance of the structures and the compactness of X imply that any two such Sobolev spaces are equivalent. A word on notation: we will write $A \lesssim B$ to mean that there exists a $C > 0$ such that $|A(u)| \le C|B(u)|$ uniformly for u in a set that will be made clear in the context.

Lemma 3.1. *Suppose that M is strongly pseudoconvex and U is an open subset of $\bar M$ with compact closure. Assume also that $\zeta, \zeta_1 \in C_c^\infty(U)$ for which $\zeta_1|_{\mathrm{supp}(\zeta)} = 1$. If $q > 0$ and $\alpha|_U \in H^s(U, \Lambda^{p,q})$, then $\zeta(\Box + 1)^{-1}\alpha \in H^{s+1}(\bar M, \Lambda^{p,q})$ and*

$$\|\zeta(\Box + 1)^{-1}\alpha\|^2_{s+1} \lesssim \|\zeta_1\alpha\|^2_s + \|\alpha\|^2_0. \tag{3.1}$$

Proof. This is Prop. 3.1.1 from [FK] extended to the noncompact case in [E]. □

It follows easily (Corollary 4.3, [P1]) that the image of the Laplacian's spectral projection P_δ is contained in $C^\infty(\bar M, \Lambda^{p,q})$.

In order to derive properties of the Schwartz kernel of P_δ, we will need global Sobolev estimates strengthening the previous result. The following assertion (Theorem 4.5 of [P1]) provides global *a priori* Sobolev estimates on M and is a generalization of Prop. 3.1.11, [FK] to the noncompact case. Note that this crucially uses the uniformity on M guaranteed by the G-action and the compactness of X.

Lemma 3.2. *Let $q > 0$. For every integer $s \ge 0$, the following estimate holds uniformly,*

$$\|u\|^2_{s+1} \lesssim \|\Box u\|^2_s + \|u\|^2_0, \qquad (u \in \mathrm{dom}\,(\Box) \cap C^\infty(\bar M, \Lambda^{p,q})).$$

The previous two lemmata give

Corollary 3.3. *For $q > 0$, let $\Box = \int_0^\infty \lambda dE_\lambda$ be the spectral decomposition of the Laplacian $\Box$ and for $\delta \ge 0$, define $P_\delta = \int_0^\delta dE_\lambda$. Then $\mathrm{im}\, P_\delta \subset H^\infty(M)$.*

Proof. The assertion follows from lemmata 3.1, 3.2 and the fact that $\operatorname{im} P_\delta \subset \operatorname{dom} \Box^k$ for all $k = 1, 2, \dots$. Thus the estimates

$$\|\Box^{k-s}u\|_{s+1} \lesssim \|\Box^{k-s+1}u\|_s + \|\Box^{k-s}u\|_0, \quad (s = 1, 2, \dots, k)$$

hold for $u \in \operatorname{im} P_\delta$. These can be reduced to the result. $\Box$

Remark 3.4. By results in [E, P3], these regularity properties essentially hold true for G-manifolds M that are weakly pseudoconvex but satisfy a subelliptic estimate. Similar results hold for the boundary Laplacian $\Box_b$ as indicated in [P1].

4. The finiteness result

In this section, we modify an ingenious lemma from [GHS]. In the original setting, this lemma asserts that on a regular covering space $\Gamma \to M \to X$, it is true that any closed, invariant subspace $L \subset L^2(M)$ that belongs to some $H^s(M)$ $(s > 0)$ has the following property. There exists an $N < \infty$ and a Γ-equivariant injection P_N such that

$$L \xhookrightarrow{P_N} L^2(\Gamma) \otimes \mathbb{C}^N.$$

This result has analogues in [A] and Theorem 8.10, [LL], gotten by different methods.

Here, we will use essentially the same proof as in [GHS] to obtain a similar result for G-bundles. We will need the following

Definition 4.1. For any positive integer s, let $H^{0,s}(G \times X) = L^2(G) \otimes H^s(X)$ be the completion of $C_c^\infty(G \times X)$ in the norm defined by

$$\|u\|^2_{H^{0,s}(G\times X)} = \int_G dt\ \|u(t,\cdot)\|^2_{H^s(X)}.$$

Clearly $\|\cdot\|_{H^{0,s}(G\times X)} \le \|\cdot\|_{H^s(M)}$ and so $H^s(M) \subset H^{0,s}(G \times X)$.

The next two statements in this section follow [GHS] closely. Lemma 4.2 is taken verbatim and Theorem 4.3 is a small variation on Prop. 1.5 of that article.

Lemma 4.2. *Let X be a compact Riemannian manifold, possibly with boundary and let $(\psi_k)_k$ be any complete orthonormal basis of $L^2(X)$. Then, for all $s > 0$ and $\delta > 0$ there exists an integer $N > 0$ such that for all $u \in H^s(X)$ in the L^2-orthogonal complement of $(\psi_k)_1^N$ we have the uniform estimate*

$$\|u\|_{L^2(X)} \le \delta \|u\|_{H^s(X)}, \qquad (u \in H^s(X),\ u \perp \psi_k,\ k = 1, 2, \dots, N).$$

Proof. Assuming the contrary, there exist $s > 0$ and $\delta > 0$ so that for each $N > 0$ there is an $u_N \in H^s(X)$ with $\langle u_N, \psi_k\rangle = 0$ for $k = 1, 2, \dots, N$ and $\|u_N\|_s < 1/\delta\ \|u_N\|_0$. Without loss of generality we may rescale the u_N to unit length. By Sobolev's compactness theorem, the sequence $(u_N)_N$ is a compact subset of $L^2(X)$. By the requirement that each u_N be orthogonal to ψ_k for $k = 1, 2, \dots, N$, the sequence converges weakly to zero. This contradicts the choice of normalization. $\Box$

Theorem 4.3. *Assume that G is a Lie group and $G \to M \to X$ is a G-bundle with compact quotient, X. Let L be an L^2-closed, G-invariant subspace in $H^\infty(M)$, such that for $s \in \mathbb{N}$ sufficiently large, $L \subset H^s(M)$ and*

$$\|u\|_{H^s(M)} \lesssim \|u\|_{L^2(M)} \tag{4.1}$$

holds uniformly for $u \in L$. Then $L \subset \operatorname{im}(\mathbf{1}_{L^2(G)} \otimes P_{L^2(X)})$ where $P_{L^2(X)}$ is a finite-rank projection in $L^2(X)$.

Proof. First, assume that $M \cong G \times X$ is a trivial bundle. For each fixed $t \in G$, define the *slice at* t, $S_t = \{(t,x) \in M \mid x \in X\}$, and note that by the trace theorem, the restrictions of functions in L to these slices are in $H^\infty(S_t)$. Note also that the invariance of L implies that all the restrictions $L|_{S_t}$ are identical. At the identity $e \in G$, choose an orthonormal basis $(\psi_j)_j$ for $L^2(S_e) \cong L^2(X)$. Let L satisfy the assumptions of the theorem and define a map $P_N : L \to L^2(G) \otimes \mathbb{C}^N$ by

$$(P_N u)(t) = (u_1(t), u_2(t), \ldots, u_N(t)),$$

where

$$u_j(t) = \langle u|_{S_t}, \psi_j \rangle_{L^2(X)}, \qquad j = 1, 2, \ldots, N.$$

We will show that P_N is injective for large N. Assume that $u \in L$ and $P_N u = 0$. The smoothness of all the structures implies that $(P_N u)(t) = 0$ identically. Lemma (4.2) and invariance imply that there is a $\delta_N > 0$ such that

$$\|u|_{S_t}\|^2_{L^2(S_t)} \le \delta_N^2 \|u|_{S_t}\|^2_{H^s(S_t)}, \quad (t \in G). \tag{4.2}$$

Integrating over $t \in G$ we obtain

$$\|u\|^2_{L^2(M)} \le \delta_N^2 \|u\|^2_{H^{0,s}(G\times X)} \le \delta_N^2 \|u\|^2_{H^s(M)}. \tag{4.3}$$

If this were possible for any N, this would contradict the estimate (4.1) unless $u = 0$, since $\delta_N \to 0$ as $N \to \infty$. To obtain the result for a trivial bundle, let N be the least integer for which P_N is injective and choose N elements $v_1, v_2, \ldots, v_N \in L$ whose restrictions to S_e are linearly independent. The result for a general bundle follows by a trivialization argument. □

Remark 4.4. We should note here that the assumptions are redundant. For L to be L^2-closed and in $H^\infty(M)$ implies the validity of an estimate (4.1) for any s.

Corollary 4.5. *Let $\Box = \int_0^\infty \lambda dE_\lambda$ be the spectral resolution of the Laplacian and for $\delta > 0$ let $P_\delta = \int_0^\delta dE_\lambda$ be a spectral projection. Also choose a piecewise smooth section $x : X \hookrightarrow M$. It follows that P_δ has a representation*

$$(P_\delta u)(t,x) = \sum_{kl=1}^{N} \int_{G\times X} dsdy\ \psi_k(x) h_{kl}(st^{-1}) \bar{\psi}_l(y) u(s,y), \tag{4.4}$$

where $(\psi_k)_k$ are an orthonormal basis of $L^2(X)$ and $H = [h_{kl}]_{kl}$ is a self-adjoint, idempotent convolution operator in $\bigoplus_1^N L^2(G)$ with $h_{kl} \in C^\infty(G)$. Also,

$$\sum_{kl=1}^{N} \|h_{kl}\|^2_{L^2_R(G)} = \sum_{k=1}^{N} h_{kk}(e) < \infty.$$

Proof. By Corollary 3.3, the theorem applies. Apply the Gram-Schmidt procedure to the $(v_k)_1^N$ above, obtaining the $(\psi_k)_1^N$. The decomposition is described in §2. □

Remark 4.6. In the case that G is unimodular, $\sum_{kl} \|h_{kl}\|^2_{L^2_R(G)} < \infty$ is the same as saying that P_δ is in the G-trace class, which we established in [P1] in the setting in which M is strongly pseudoconvex and in [P3] where M satisfies a subelliptic estimate. The new content of Corollary 4.5 is the finiteness of the sum (4.4), *etc.* This transverse dimension gives a meaningful (though much rougher) measure of the spectral subspaces of $\Box$ (and $\Box_b$) than the G-dimension when G is unimodular, but is also defined when the group is not assumed unimodular as, for example, in [HHK] and in important examples in [GHS]. We should note that [HHK] also

deals with the situation in which the G-action is only proper, rather than free as we assume here.

5. Applications

We will give a version of the solution of the $\bar{\partial}$-Neumann problem, for our noncompact M. The version valid for M compact, *e.g.* Prop. 3.1.15 of [FK], is unlikely to remain valid in our setting because the Neumann operator on a noncompact space is usually unbounded.

Let $\Box = \int_0^\infty \lambda dE_\lambda$ be the spectral decomposition of the Laplacian on M and for $\delta > 0$ put

$$L_\delta = \operatorname{im} \int_\delta^\infty dE_\lambda \quad \text{and} \quad P_\delta = \int_0^\delta dE_\lambda. \tag{5.1}$$

In this section we will show that $\Box u = g$, and the $\bar{\partial}$-Neumann problem have regular solutions for $g \in L_\delta$.

Lemma 5.1. *If $g \in L_\delta \cap C^\infty(\bar{M})$, then the solution u of $\Box u = g$ is smooth.*

Proof. Let $g \in L_\delta \cap C^\infty(\bar{M})$ and solve $\Box u = g$ in $L^2(M)$. Note that $\|u\|_{L^2(M)} \leq (1/\delta)\|g\|_{L^2(M)}$. Adding u to both sides of the equation, $(\Box + 1)u = g + u$, we obtain that $(\Box + 1)u = \Box u + u = g + u$. Applying $(\Box + 1)^{-1}$, the real estimate, Lemma 3.1 provides that

$$\|\zeta u\|_{s+1} \lesssim \|\zeta_1(g+u)\|_s + \|g+u\|_0 \leq \|\zeta_1 g\|_s + \|\zeta_1 u\|_s + \|g+u\|_0.$$

Nesting the supports of cutoff functions, concatenating and reducing these estimates for $s = 0, 1, \ldots$, we obtain that for each positive integer s we have

$$\|\zeta u\|_{s+1} \lesssim \|\zeta_1 g\|_s + \|g+u\|_0 \leq \|\zeta_1 g\|_s + (1 + 1/\delta)\, \|g\|_0.$$

Thus $u \in C^\infty(\bar{M})$ by the Sobolev embedding theorem. $\Box$

Corollary 5.2. *In L_δ, the Laplacian satisfies the genuine estimate*

$$\|u\|_{s+1} \lesssim \|\Box u\|_s + \|u\|_0, \qquad (u \in L_\delta).$$

Proof. Let $(g_k)_k \subset L_\delta \cap H^\infty$ and $g_k \to g \in H^s(M)$. The previous lemma implies that there exists a sequence $(u_k)_k \subset C^\infty$ solving $\Box u_k = g_k$. Lemma 3.2 implies that $\|u_k\|_{s+1} \lesssim \|\Box u_k\|_s + \|u_k\|_0$ uniformly in k, so $(u_k)_k$ is Cauchy in the H^{s+1} norm. $\Box$

Lemma 5.3. *Suppose that $q > 0$, $\alpha \in L^2(M, \Lambda^{p,q}), \bar{\partial}\alpha = 0$, and $\alpha \in L_\delta$. Then there is a unique solution ϕ of $\bar{\partial}\phi = \alpha$ with $\phi \perp \ker(\bar{\partial})$. If $\alpha \in H^s(\bar{M}, \Lambda^{p,q})$, then $\phi \in H^s(\bar{M}, \Lambda^{p,q-1})$ and $\|\phi\|_s \lesssim \|\alpha\|_s$ for each s.*

Proof. Taking $\alpha \in L_\delta$, there is a unique solution to $\Box u = \alpha$ orthogonal to the kernel of $\Box$; in fact $u \in L_\delta \subset (\ker \Box)^\perp$. Since $\bar{\partial}\alpha = 0$, applying $\bar{\partial}$ to

$$\Box u = \bar{\partial}^* \bar{\partial} u + \bar{\partial}\bar{\partial}^* u = \alpha$$

gives that $\bar{\partial}\bar{\partial}^*\bar{\partial} u = 0$. This implies that $\langle \bar{\partial}\bar{\partial}^*\bar{\partial} u, \bar{\partial} u\rangle = 0$ which is equivalent to $\|\bar{\partial}^*\bar{\partial} u\|^2 = 0$. Thus $\bar{\partial}\bar{\partial}^* u = \alpha$ and we may take $\phi = \bar{\partial}^* u \in \operatorname{im} \bar{\partial}^*$. But $\operatorname{im} \bar{\partial}^* \subset (\ker \bar{\partial})^\perp$. The regularity claim follows immediately from Corollary 5.2 and the order of $\bar{\partial}^*$. $\Box$

Putting all these results together, we obtain

Corollary 5.4. *Let M be a complex manifold on which a subelliptic estimate holds. Assume also that M is the total space of a bundle $G \to M \to X$ with G a Lie group acting by holomorphic transformations with compact quotient $X = M/G$. With respect to a piecewise smooth section $X \hookrightarrow M$, define the slices S_t. Then there exists a finite-dimensional subspace $L|_{S_e} \subset L^2(X)$, such that the equation $\Box u = \alpha$ has solutions $u \in L^2(M)$ with uniform estimates on the space of α satisfying $\alpha|_{S_t} \perp L|_{S_e}$ for all $t \in G$.*

Proof. Choose $\delta > 0$. Corollary 3.3 and Theorem 4.3 imply that there exists a finite rank projection $P_{L^2(X)} \in \mathcal{B}(L^2(X))$ such that $P_\delta < \mathbf{1}_{L^2(G)} \otimes P_{L^2(X)}$. The orthogonal complement of the latter projection is $\mathbf{1}_{L^2(G)} \otimes P^\perp_{L^2(X)}$, which contains L_δ, on which the $\bar\partial$-Neumann problem is regular by the results of this section. Putting $L|_{S_e} = \operatorname{im} P_{L^2(X)}$, we have the result. $\square$

Remark 5.5. A similar result holds for the $\bar\partial$-equation by Lemma 5.3.

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